

Asymptotic flatness at null infinity in higher dimensional gravity

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Abstract

We give a geometrical definition of the asymptotic flatness at null infinity in spacetimes of even dimension d greater than 4 within the framework of conformal infinity. Our definition is shown to be stable against perturbations to linear order. We also show that our definition is stringent enough to allow one to define the total energy of the system viewed from null infinity as the generator conjugate to an asymptotic time translation. We derive an expression for the generator conjugate within the Hamiltonian framework, and propose to take this notion of energy as the natural generalisation of the Bondi energy to higher dimensions. Our definitions of asymptotic flatness and the Bondi energy formula differ qualitatively from the corresponding definitions in $d = 4$; although the asymptotic structure of null infinity in higher dimensions parallels that in 4-dimensions in some ways, the latter seems to be a rather special case on the whole compared to general $d > 4$. Our definitions and constructions do not work in odd spacetime dimensions, essentially because the unphysical metric seems to have insufficient regularity properties at null infinity in that case.

1 Introduction

Gravity in higher dimensions has become one of the major subjects of recent studies in fundamental physics. There are a lot of questions that have been already answered in the 4-dimensional case but remain open in higher-dimensions. Amongst them, perhaps the most fundamental issue is how to define the notion of an *isolated system* and associated *conserved*

quantities in higher-dimensions. The purpose of this note¹ is to answer this question, providing a definition of *asymptotic flatness at null infinity in higher-dimensions*.

In general relativity, the idea of an isolated system is given by defining “asymptotic flatness” at infinity². One can consider two different infinities where the spacetime curvature vanishes: a spatial infinity and a null infinity. Accordingly, in 4-dimensions, one has two notions of total gravitational energies: the ADM-energy [2] defined at a spatial infinity and the Bondi-energy [3] measured at a null infinity. The ADM-energy is constant. Since it is essentially defined by inspecting the behaviour of the Coulomb part of Weyl components of gravity at large spatial distances, it is straightforward to generalise the definition of the ADM-energy to the higher-dimensional case. On the other hand, the Bondi-energy is in general a function of time in the sense that it depends on the chosen cross section at null infinity. The difference between Bondi-energies measured at two different times represents the flux of gravitational radiation through the portion of null infinity bounded by the corresponding two cross sections. Since in higher-dimensions, the behaviour of the radiating part of gravitation is different from that of the Coulomb part, it is not a trivial matter to generalise the definition of the Bondi-energy to the higher-dimensional case.

In order to get a sensible definition of a higher-dimensional version of the Bondi-energy, we first provide an appropriate generalisation of asymptotic flatness to higher-dimensions. Such a definition of asymptotic flatness should be arrived at by inspecting the fall off behaviour of gravitational perturbations near null infinity, so that the definition will be stable under at least linear perturbations. Actually, as we will see, in d -dimensions, perturbations typically drop off as $1/r^{(d-2)/2}$ as one approaches null infinity, which differs from the drop off rate of the Schwarzschild metric $1/r^{d-3}$ in higher dimensions $d > 4$. Consequently our definition of asymptotic flatness in $d > 4$ dimensions differs qualitatively from that in 4-dimensions.

We then derive an expression for the generator conjugate to an asymptotic time translation symmetry for asymptotically flat spacetimes in d -dimensional general relativity (d even) within the Hamiltonian framework, making use especially of a formalism developed by Wald and Zoupas [4]. This generator is given by an integral over a cross section at null infinity of a certain local expression and is taken to be the definition of the Bondi-energy in d -dimensions. Our definition yields a manifestly positive flux of radiated energy.

¹This article is a concise version of the paper [1].

²Roughly speaking, an isolated system in general relativity is a spacetime that looks like Minkowski spacetime far away in “any directions along spacelike or null curves.” Other, less restrictive notions of an isolated system may also be considered, for example systems that look like a Kaluza-Klein space far out in the “non-compact directions.” However, the analysis of such metrics and of the associated conserved quantities would be substantially different from the ones studied here.

2 Asymptotic flatness at null infinity

Asymptotic conditions in field theory require the specification of a background configuration and the precise rate at which this background is approached. In the case of asymptotic flatness in higher dimensional general relativity, a natural background is the Minkowski metric. Let $(\tilde{M}, \tilde{g}_{ab})$ be a d -dimensional physical spacetime. We are concerned with how to specify the precise rate at which Minkowski space is approached $(\tilde{M}, \tilde{g}_{ab})$ at large distances in null directions.

Unphysical spacetime (M, g_{ab}) : It is of great technical advantage to work within a framework in which “infinity” \mathcal{I} is attached as additional points to a physical spacetime manifold, \tilde{M} . One obtains an “unphysical spacetime manifold” with boundary,

$$M \equiv \tilde{M} \cup \mathcal{I}.$$

Furthermore, the points at infinity are brought metrically to a finite distance by rescaling the physical metric, \tilde{g}_{ab} , by a conformal factor Ω^2 with suitable properties. The asymptotic flatness conditions are then formulated in terms of this rescaled “unphysical metric,”

$$g_{ab} \equiv \Omega^2 \tilde{g}_{ab}, \quad (1)$$

and its relation to the likewise conformally rescaled version of Minkowski space, to which we will refer as the “background geometry.”

Background geometry (\bar{M}, \bar{g}_{ab}) : For definiteness, we take \bar{M} to be the region $\{-\pi/2 \leq t \pm \psi \leq \pi/2\}$ of $\mathbb{R} \times S^{d-1}$, where t is the coordinate of \mathbb{R} and ψ is the azimuthal angle of S^{d-1} . We take the metric \bar{g}_{ab} to be the line element of the Einstein static universe, $ds^2 = -dt^2 + d\psi^2 + \sin^2 \psi d\sigma^2$. As it is well-known, the metric of the Einstein static universe is related to the Minkowski metric $\tilde{\eta}_{ab} = -dx_0^2 + dx_1^2 + \dots + dx_{d-1}^2$ by

$$\bar{g}_{ab} = \Omega^2 \tilde{\eta}_{ab}, \quad (2)$$

where $\Omega = \cos(\psi - t) \cos(\psi + t)$, and Minkowski spacetime corresponds precisely to the region \bar{M} . The boundary of \bar{M} are the conformal infinities of Minkowski spacetime.

Tensor fields on Minkowski spacetime can be identified with tensor fields on \bar{M} , and their rate of decay at null infinity can be considered. To have a quantitative notion, we make the following definition.

Definition: A tensor field, $L_{ab\dots c}$, is said to be of order Ω^s with $s \in \mathbb{R}$, written $L_{ab\dots c} = O(\Omega^s)$, if the tensor field $\Omega^{-s} L_{ab\dots c}$ is smooth at the boundary of \bar{M} .

Consequently, if $L_{ab\dots c}$ is of order s , then $\Omega^r L_{ab\dots c}$ is of order $s + r$, and $\bar{\nabla}_{d_1} \dots \bar{\nabla}_{d_k} L_{ab\dots c}$ is of order $s - k$.

2.1 Definition of asymptotic flatness

We now state our definition of asymptotic flatness in even spacetime dimensions $d > 4$.

Definition (Asymptotic flatness): Let $(\tilde{M}, \tilde{g}_{ab})$, (M, g_{ab}) , and (\bar{M}, \bar{g}_{ab}) be, respectively, a d -dimensional spacetime, an unphysical spacetime, and a background spacetime defined above, with smooth conformal factor Ω . A spacetime $(\tilde{M}, \tilde{g}_{ab})$ is said to be *weakly asymptotically simple at null infinity* if the following is true:

1. It is possible to attach a boundary, \mathcal{I} , to \tilde{M} such that there exists an open neighbourhood of \mathcal{I} in $M = \tilde{M} \cup \mathcal{I}$ which is diffeomorphic to an open subset of the manifold \bar{M} of our background geometry, such that \mathcal{I} gets mapped to a subset of the boundary of \bar{M} under this identification.
2. One has, relative to our background metric \bar{g}_{ab} , that

$$\bar{g}_{ab} - \tilde{g}_{ab} = O(\Omega^{\frac{d-2}{2}}), \quad \bar{\epsilon}_{ab\dots c} - \epsilon_{ab\dots c} = O(\Omega^{\frac{d}{2}}), \quad (3)$$

where $\bar{\epsilon}_{ab\dots c}$ and $\epsilon_{ab\dots c}$ denote the volume element (viewed as d -forms) associated with the metrics \bar{g}_{ab} respectively g_{ab} , as well as

$$(\bar{g}^{ab} - g^{ab})(d\Omega)_a = O(\Omega^{\frac{d}{2}}), \quad (\bar{g}^{ab} - g^{ab})(d\Omega)_a(d\Omega)_b = O(\Omega^{\frac{d+2}{2}}), \quad (4)$$

where g^{ab} is the inverse of g_{ab} and where \bar{g}^{ab} is the inverse³ of \bar{g}_{ab} .

Remarks: (i) The physical metric remains unchanged if we change the background metric to $\bar{g}'_{ab} = k^2 \bar{g}_{ab}$ and the conformal factor to $\Omega' = k\Omega$, with k non-vanishing and smooth at the boundary of \tilde{M} . It is easily seen that our definition of an asymptotically flat spacetime is independent under such a change of “conformal gauge.” Hence, in this sense, our definition is independent of the particular background geometry chosen.

(ii) As in 4-spacetime dimensions, the notion of weak asymptotic simplicity can be strengthened by requiring in addition that every inextendible null geodesic in $(\tilde{M}, \tilde{g}_{ab})$ has precisely two endpoints on \mathcal{I} . Such a spacetime is then simply called *asymptotically simple*. This additional condition, combined with the fact that \mathcal{I} is null, makes it possible to divide \mathcal{I} into disjoint sets, \mathcal{I}^+ and \mathcal{I}^- , on which future respectively past directed null geodesics have their endpoints. These sets are referred to as future respectively past null infinity. This condition also implies that $(\tilde{M}, \tilde{g}_{ab})$ necessarily has to be globally hyperbolic, by a straightforward generalisation of Prop. 6.9.2 of [6] to d -dimensions.

(iii) Let us compare the above definition of asymptotic flatness with the behaviour of the d -dimensional Schwarzschild metric, given by the line element

$$d\tilde{s}^2 = -\left(1 - \frac{c}{r^{d-3}}\right) dt^2 + \left(1 - \frac{c}{r^{d-3}}\right)^{-1} dr^2 + r^2 d\sigma^2, \quad c > 0. \quad (5)$$

Introducing a coordinate u by the relation $du = dt - (1 - cr^{-(d-3)})^{-1} dr$, the line element takes the form

$$d\tilde{s}^2 = -2dudr - du^2 + r^2 d\sigma^2 + cr^{-(d-3)} du^2. \quad (6)$$

³We use the convention that indices on tensors with a tilde are raised and lowered with \tilde{g}_{ab} , and those with a “bar” are raised and lowered with \bar{g} , and those without are with g_{ab} .

The first three terms on the right side are recognised as the Minkowski line element. Multiplying by our conformal factor Ω^2 , using $r^{-1} = O(\Omega)$, and using that Ω^2 times the Minkowski metric is equal to our background metric $d\bar{s}^2$ by construction, it follows that the unphysical Schwarzschild metric can be written as

$$ds^2 - d\bar{s}^2 = O(\Omega^{d-1})du^2$$

(noting that u is a good coordinate at infinity). It follows that Schwarzschild spacetime is asymptotically flat in the sense of our definition, but it becomes flat at null infinity at a faster rate than that specified above in conditions given in eqs. (3) and (4) in $d > 4$. [In $d = 4$, the relevant components drop off at the same rate, see the last in eqs. (4).]

(iv) The above definition of asymptotic flatness is not appropriate in odd spacetime dimension, since condition (3) in item 2 now says that the unphysical metric g_{ab} differs from the smooth background metric \bar{g}_{ab} by a half odd integer power of Ω , and thereby manifestly contradicts the assumption in item 1 that g_{ab} is smooth at the boundary. The powers of Ω appearing in eqs. (3) and (4) reflect the drop off behaviour of a linearised perturbation (see in the next section), and it is hard to see how these powers could be any different from the ones in the full nonlinear theory. It therefore appears that the unphysical metric is generically at most $(d-3)/2$ times differentiable at the boundary in odd dimensions. We note that it is also inconsistent in odd dimensions to postulate that the quantity $\Omega^{-(d-2)/2}(g_{ab} - \bar{g}_{ab})$ is smooth at the boundary as we did above in eq. (3) of item 2 in the even dimensional case, because the unphysical Schwarzschild metric g_{ab} differs from the background \bar{g}_{ab} by terms of order Ω^{d-1} , i.e., by an even power of Ω . Therefore, eq. (3) is definitely false for the Schwarzschild metric in odd dimensions. For the Schwarzschild metric, $\Omega^{-(d-1)}(g_{ab} - \bar{g}_{ab})$ is smooth at the boundary (in even and odd dimensions), so one might be tempted to try this condition, together with suitable other conditions, as the definition of asymptotic flatness. However, this would eliminate from consideration all radiating spacetimes and is therefore not acceptable. One may try to bypass these problems by requiring appropriate lower differentiability properties of the corresponding quantities, but these seem neither to lead to a definition of asymptotic flatness that is stable under perturbations, nor do those weaker conditions seem to be able to guarantee the existence of conserved quantities such as the Bondi-energy.⁴ Thus, it seems that a sensible definition of asymptotic simplicity at conformal infinity in odd spacetime dimensions would have to differ substantially from the one given above for even dimensions, and it is doubtful that such a definition can be cast into the framework of conformal infinity.

(v) We finally comment on how the above definition of asymptotic flatness in even spacetime dimensions $d > 4$ compares to the usual definition [8] in 4-dimensions. In this definition, one simply demands that there

⁴Such a difficulty in defining some quantities associated with radiations in odd spacetime dimensions reminds us of the well-known fact that in odd-dimensions the manner of radiation propagation is qualitatively different from that in even-dimensions (see e.g., [7] and references therein).

exists *some* conformal factor, Ω , such that the corresponding unphysical metric is smooth at \mathcal{I} and such that n_a is non-vanishing and null there. Note that the nullness of n_a follows from the first condition if Einstein's equation with vanishing stress energy at null infinity are assumed. This definition is different in appearance from that given above and avoids in particular the introduction of a background geometry. Nevertheless, the definition of asymptotic flatness in $d = 4$ as just stated can be brought⁵ into a form that is very similar (but not identical) to the one given above for $d > 4$. To see this in more detail, we recall that the usual definition of asymptotic flatness in 4-dimensions is equivalent [9] to the statement that with the vacuum Einstein equations, the physical metric can be cast into the “Bondi form” (see eqs.(14) and (31)–(34) of [3]),

$$\begin{aligned} d\tilde{s}^2 = & -2dudr - du^2 + r^2 d\sigma^2 \\ & + O(r)d(\text{angles})^2 + O(1)dud(\text{angles}) \\ & + O(r^{-1})du^2 + O(r^{-2})dudr \end{aligned} \quad (7)$$

in suitable coordinates near null infinity, where the first line is recognised as the Minkowski line element (with $d\sigma^2$ the line element of the unit 2-sphere).

In $d > 4$ spacetime dimensions our asymptotic flatness conditions in effect state that the physical line element can be written in the form

$$\begin{aligned} d\tilde{s}^2 = & -2dudr - du^2 + r^2 d\sigma^2 \\ & + O(r^{-\frac{d-4}{2}})d(\text{angles})^2 + O(r^{-\frac{d-4}{2}})dud(\text{angles}) \\ & + O(r^{-\frac{d-2}{2}})du^2 + O(r^{-\frac{d}{2}})dudr, \end{aligned} \quad (8)$$

where “angles” now stands for the polar angles of S^{d-2} , and $d\sigma^2$ is the line element on S^{d-2} . The Bondi form (8) in $d > 4$ does not reduce to eq. (7) when d is set to 4. The difference between the two expressions arises from the $d(\text{angles})^2$ -term, which quantifies the perturbations in the size of the cross sections of a lightcone relative to Minkowski spacetime. According to eq. (7), this term is of order $O(1)$ in $d = 4$ for a radiating metric, whereas eq. (8) would say that it ought to be of order $O(r^{-1})$. The latter is simply wrong for a radiating metric in 4 dimensions. This difference can be traced back to the last of conditions (3) in $d > 4$ dimensions, which therefore does not hold in $d = 4$. This special feature of 4 dimensions will be reflected in corresponding differences in our discussion of the Bondi energy in dimensions $d > 4$. We will therefore, for the rest of this paper, keep the case $d = 4$ separate and assume throughout that $d > 4$ (and even). Our formulas will not be valid in $d = 4$ unless stated otherwise.

Asymptotic symmetry: A diffeomorphism ϕ is called an *asymptotic symmetry* if it transforms any asymptotically flat metric to an asymptotically flat metric. The asymptotic symmetries form a group under the

⁵We emphasise, however, that an analogous statement is not true in $d > 4$. Namely, it is not true that our definition of asymptotic flatness in higher-dimensions is equivalent to the statement that there exists some conformal factor, Ω , such that the corresponding unphysical metric is smooth at \mathcal{I} and such that $(d\Omega)_a$ is non-vanishing and null there.

composition of two diffeomorphisms. An infinitesimal asymptotic symmetry is a smooth vector field ξ^a on \tilde{M} that has a smooth extension (denoted by the same symbol) to the unphysical manifold, M , and which generates a 1-parameter group of asymptotic symmetries. It is a direct consequence of our definitions that the quantity

$$\chi_{ab} \equiv \Omega^{-\frac{d-6}{2}} \mathcal{L}_\xi \tilde{g}_{ab}$$

has to satisfy

$$\begin{aligned} \chi_{ab} &= O(1), \quad \chi_a{}^a = O(\Omega), \quad \chi_{ab} n^a = O(\Omega), \quad \chi_{ab} n^a n^b = O(\Omega^2), \\ \xi^a n_a &= O(\Omega), \end{aligned} \quad (9)$$

where here and in the following,

$$n_a \equiv (d\Omega)_a.$$

Conversely, if the above relations are satisfied for some asymptotically flat spacetime, then ξ^a is an infinitesimal asymptotic symmetry.

It turns out [1] that in higher dimensions, there is no direct analog of the infinite set of angle dependent translational symmetries known in 4-dimensions.

One can use the freedom to choose any metric in the conformal equivalence class of the Einstein static universe (see (i)) to make convenient “conformal gauge choices.” A particularly useful choice for many purposes is one for which

$$\bar{\nabla}_a (d\Omega)_b = 0, \quad \bar{g}^{ab} (d\Omega)_a (d\Omega)_b = 0 \quad (10)$$

in an open neighbourhood of the boundary, where \bar{g}^{ab} is the inverse metric and $\bar{\nabla}_a$ the derivative operator associated with the background metric. Our formula for the Bondi energy and flux assume this gauge.

2.2 Stability of asymptotic flatness

We justify our definition of asymptotic flatness for even d given above, by showing that the definition is stable under linear perturbations.

Theorem (Stability of asymptotic flatness): Let $(\tilde{M}, \tilde{g}_{ab})$ be a globally hyperbolic solution to the vacuum Einstein equations $\tilde{R}_{ab} = 0$. Consider a solution $\delta\tilde{g}_{ab}$ to $\delta\tilde{R}_{ab} = 0$ whose initial data have compact support on a Cauchy surface. Then, there exists a gauge such that setting $\delta g_{ab} = \Omega^2 \delta\tilde{g}_{ab}$,

$$\begin{aligned} \delta g_{ab} &= O(\Omega^{\frac{d-2}{2}}), \quad \delta g_{ab} n^a = O(\Omega^{\frac{d}{2}}), \\ \delta g_{ab} n^a n^b &= O(\Omega^{\frac{d+2}{2}}), \quad g^{ab} \delta g_{ab} = O(\Omega^{\frac{d}{2}}), \end{aligned} \quad (11)$$

at \mathcal{I} for all even $d > 4$.

Note that these conditions are the linearised version of our definition of asymptotic flatness, eqs. (3) and (4), about an asymptotically flat background. Our definition of asymptotic flatness is therefore stable to linear order.

Sketch of proof: We choose the transverse-traceless gauge

$$\tilde{\nabla}^a \delta \tilde{g}_{ab} = \tilde{g}^{ab} \delta \tilde{g}_{ab} = 0, \quad (12)$$

and define field variables by

$$\phi_\alpha = \begin{cases} \tau_{ab} \equiv \Omega^{-(d-2)/2} \delta g_{ab} \\ \tau_a \equiv \Omega^{-1} \tau_{ab} n^b \\ u \equiv \nabla^a \tau_a \end{cases}. \quad (13)$$

Then we can reduce the linearised Einstein equations to the form

$$\nabla^a \nabla_a \phi_\alpha = A_\alpha^{\beta a} \nabla_a \phi_\beta + B_\alpha^\beta \phi_\beta, \quad (14)$$

where $A_\alpha^{\beta a}$ and B_α^β are smooth tensor fields up to and on \mathcal{I} . Since the hyperbolic system (14) possesses a well-defined initial value formulation [5] in the unphysical spacetime, we have a smooth extension of ϕ_α on \mathcal{I} . \square (For details, see Ref. [1].)

Remarks: (i) In 4-dimensions, the corresponding theorem was proved by Geroch and Xanthopoulos [10], in which neither the transverse-traceless gauge nor our above choice of the field variables work. Instead, it is necessary to work in the so called Geroch-Xanthopoulos gauge, and to take other field variables given in ref. [10]. That analysis shows that the fall off rate of the perturbation are

$$\begin{aligned} \delta g_{ab} &= O(\Omega), \quad \delta g_{ab} n^b = O(\Omega^2), \\ \delta g_{ab} n^a n^b &= O(\Omega^3), \quad g^{ab} \delta g_{ab} = O(\Omega) \end{aligned} \quad (15)$$

in 4-dimensions. We notice that the trace of the perturbation is falling off as fast as the metric perturbation itself. This property differs from that in the higher-dimensional case, where the trace falls off one power faster [see the last of eqs. (11)].

(ii) In odd d case, since g_{ab} itself is not smooth at \mathcal{I} , the coefficients $A_\alpha^{\beta a}$ and B_α^β are no longer smooth at \mathcal{I} , hence one cannot guarantee the existence of smooth solutions to the hyperbolic system (14).

3 Gravitational energy at null infinity

We now define the Bondi-energy in higher-dimensions, using the general strategy by Wald and Zoupas [4] for defining charges associated with “boundaries” in theories derived from a general diffeomorphism covariant Lagrangian. For the case of Einstein gravity, $L = (1/16\pi G) \tilde{R} \tilde{\epsilon}$, with the boundary given by null infinity \mathcal{I} , their scheme is as follows. Let θ be the $(d-1)$ -form defined by $\delta L = \text{Einstein's equation} + d\theta$, and let ω be the symplectic current $\omega(\delta_1 g, \delta_2 g) = \delta_1 \theta(\delta_2 \tilde{g}) - \delta_2 \theta(\delta_1 \tilde{g})$. Further, let ξ^a be a vector field on representing an infinitesimal asymptotic symmetry. If one can show that

- (1) ω has a well-defined (finite) extension to the \mathcal{I} for any asymptotically flat metric,

(2) there exists a symplectic potential Θ on \mathcal{I} such that

$$(\text{pullback of } \omega \text{ to } \mathcal{I}) = \delta_1 \Theta(\delta_2 g) - \delta_2 \Theta(\delta_1 g),$$

then the Wald-Zoupas method ensures that one can define an associated charge \mathcal{H}_ξ by

$$\delta \mathcal{H}_\xi = \int_B (\delta Q_\xi - \xi \cdot \theta) + \int_B \xi \cdot \Theta, \quad (16)$$

where

$$Q_\xi = -\frac{1}{16\pi G}(\tilde{\nabla}\xi) \cdot \tilde{\epsilon}$$

is the Noether charge $(d-2)$ -form, and B a given cross section at \mathcal{I} .

Note that it is not immediately evident that the above equation actually defines a charge \mathcal{H}_ξ (up to an arbitrary constant), i.e., that the right side of eq. (16) is indeed the “ δ ” of some quantity. To see this, one first verifies that the right side of eq. (16) has a vanishing anti-symmetrized second variation⁶. This is certainly a necessary condition for it to arise as the “ δ ” of some quantity \mathcal{H}_ξ , for we always have $(\delta_1 \delta_2 - \delta_2 \delta_1) \mathcal{H}_\xi = 0$. As argued in [4], this is also a sufficient condition if one assumes that the space of asymptotically flat metrics is simply connected. The arbitrary constant is fixed by setting \mathcal{H}_ξ equal to zero on Minkowski spacetime.

One can show that the assumptions (1) and (2) that are needed for the existence of \mathcal{H}_ξ indeed hold under our choice of the boundary conditions (fall off conditions) for asymptotic flatness. Namely, one can show that the pullback of ω to \mathcal{I} can be written in terms of the smooth variables τ_{ab}, τ_a at \mathcal{I} , and that Θ can be given in terms of a smooth tensor field at \mathcal{I} with vanishing trace, called the

News tensor in higher-dimensions:

$$N_{ab} \equiv \text{pullback to } \mathcal{I} \text{ of } \Omega^{-(d-4)/2} q^m{}_a q^n{}_b S_{mn},$$

where $q_{ab} \equiv g_{ab} - 2n_{(a} l_{b)}$, with $l_a l^a = 0$, $n^a l_a = +1$, is the projection onto \mathcal{I} and S_{ab} is defined by $(d-1)(d-2)S_{ab} \equiv 2(d-1)R_{ab} - Rg_{ab}$. In fact, Θ is expressed as⁷ $\Theta_{a_1 \dots a_{d-1}} = (1/32\pi G)\tau^{cd} N_{cd} \epsilon_{a_1 \dots a_{d-1}}$.

If ξ^a is *asymptotic time translation*, then it can be written as $\xi^a = \alpha n^a - \Omega \nabla_a \alpha$ with α a suitable function that specifies the translation in question. We now restrict our consideration to the special case of such translation asymptotic symmetries. Then, as the explicit expression of \mathcal{H}_ξ , we obtain

⁶This would not be so if we had not added the Θ -term to the expression for $\delta \mathcal{H}_\xi$.

⁷Although Θ has a “gauge freedom” with respect to the change of the conformal factor Ω , this conformal gauge freedom can actually be fixed by imposing the gauge condition (10) on the background metric \bar{g}_{ab} [1], which is seen to yield the following conditions on the unphysical metric,

$$n^a n_a = O(\Omega^{(d+2)/2}), \quad \nabla_a n_b = O(\Omega^{(d-2)/2}).$$

Our results for the Bondi energy formula are obtained under this gauge fixing.

Bondi-energy formula in even dimensions $d > 4$: For any such infinitesimal translation, the Bondi energy (-momentum) is given by

$$\begin{aligned} \mathcal{H}_\xi = & \frac{1}{8(d-3)\pi G} \int_B \Omega^{-(d-4)} \left(\frac{1}{(d-2)} R_{ab} q^{ac} q^{bd} (\nabla_c l_d) \xi^e l^f \right. \\ & \left. - \Omega^{-1} \alpha^{-1} (l^{[e} - v \nabla^{[e} \log \alpha) C^{f]bcd} \xi_b (l_c - v \nabla_c \log \alpha) \xi_d \right) \epsilon_{ef a_1 \dots a_{d-2}}, \end{aligned} \quad (17)$$

where v is defined by $\nabla_a v = l_a$, and $\epsilon_{ef a_1 \dots a_{d-2}}$ denotes the natural volume element on (M, g_{ab}) , and where we are assuming that α is such that ξ^a corresponds to a null-translation to keep the formula simple.

We also have the flux formula associated with such a ξ^a through a segment \mathcal{S} of \mathcal{I}

$$F_\xi = -\frac{1}{32\pi G} \int_{\mathcal{S}} \alpha N^{cd} N_{cd} {}^{(d-1)}\epsilon. \quad (18)$$

For positive α , ξ^a is future directed timelike or null at null-infinity. This shows that the flux of energy (defined via *any* asymptotic future-directed translational symmetry) through \mathcal{S} is always negative, i.e., that the energy radiated away by the system is always positive.

In the case $d = 4$, the energy formula (17) is not correct and needs to be modified by replacing $1/(d-2)R_{ab}$ by the combination $(1/2)(S_{ab} - \rho_{ab})$, see [4]. It then coincides with an expression for the quantities associated with asymptotic translations first proposed by Geroch [8].

The first and second term in the integrand of (17) can be roughly interpreted as follows: the second term is the ‘‘Coulomb part’’ of the Weyl tensor (multiplied by suitable powers of Ω), and represents the ‘‘pure Coulomb contribution’’ to the Bondi energy. The first term represents contributions from gravitational radiation; it follows from the conditions for the vector field l^a that it vanishes if and only if the news tensor, N_{ab} , and hence the flux, vanishes. In 4-dimensions, it can be proved [8] that the news tensor, and hence the radiative contribution to the Bondi energy, always vanishes in stationary spacetimes. It would be interesting to see whether an analog of this result holds in d -dimensions.

In the d -dimensional Schwarzschild spacetime, the Bondi energy is evaluated as follows. The term involving R_{ab} in our expression (17) for the Bondi energy does not contribute, showing that there is no radiative contribution to the Bondi energy. The Coulomb contribution is found to be $\Omega^{-(d-3)} C^{abcd} l_a n_b l_c n_d = c(d-2)(d-3)/4$ at \mathcal{I} . Normalising ξ^a so that it coincides with the timelike Killing field t^a of the metric (5). Inserting this into our Bondi-energy formula, we get

$$\mathcal{H}_{\alpha n} = \frac{c(d-2)A_{d-2}}{16\pi G} \quad (= \frac{c}{2G} \text{ in 4-dimensions}). \quad (19)$$

This coincides with the ADM mass of the spacetime (5) (given e.g. in [11]), as we expect.

The Bondi-energy at a cross section $B \subset \mathcal{I}$ is interpreted as, naively speaking, the ADM-energy minus the energy of gravitational radiation

emitted by some isolated system in the causal past of B . If the Bondi-energy of an isolated system became negative, that would imply that the system radiates away more energy than it had initially. It is not possible to tell from the above integral expression if the Bondi-energy is positive or not, and therefore a further analysis is needed. In the 4-dimensional case, positivity was confirmed in [12, 13, 14], but for higher-dimensions, this issue is still open.

References

- [1] Hollands, S. and Ishibashi, A.: gr-qc/0304054.
- [2] Arnowitt, R., Deser, S. and Misner, C. W.: in *Gravitation: An introduction to current research*, ed. L. Witten, Wiley, New York (1962)
- [3] Bondi, H., van der Burg, M. G. J. and Metzner, A. W. K.: *Proc. R. Soc. London* **A269**, 21-52 (1962)
- [4] Wald, R. M. and Zoupas, A.: *Phys. Rev.* **D61**, 084027 (2000)
- [5] Wald, R. M.: *General Relativity*, University of Chicago Press, Chicago (1984)
- [6] Hawking, S. W. and Ellis, G. F. R.: *The large scale structure of space-time* (Cambridge University Press, Cambridge, 1973)
- [7] Cardoso, V., Dias, O. J. C. and Lemos, J. P. S.: *Phys. Rev. D* **67**, 064026 (2003)
- [8] Geroch, R.: in *Asymptotic structure of spacetime*, ed. F. Esposito and L. Witten, Plenum Press, New York (1977)
- [9] Tamburino, L. and Winicour, J.: *Phys. Rev.* **150**, 1039 (1966)
- [10] Geroch, R. and Xanthopoulos, B. C.: *J. Math. Phys.* **19**, 714 (1978)
- [11] Myers, R. C. and Perry, M. J.: *Annals Phys.* **172**, 304 (1986)
- [12] Ludvigsen, M. and Vickers, J. A. G.: *J. Phys. A Math. Gen.* **14** L389 (1981)
- [13] Schoen, R. and Yau, S.-T.: *Phys. Rev. Lett.* **48** 369 (1982)
- [14] Horowitz, G. T. and Perry, M. J.: *Phys. Rev. Lett.* **48** 371 (1982)